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The symmetries of slow motions and the equilibria of continua ☆

O.V. Voinov
Tyumen, Russia

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Abstract

The spatial fields of the velocities of slow flows of a viscous incompressible fluid and the equilibrium displacements in elastic media are considered within the framework of models of continua which satisfy the Navier-Stokes or Hooke's laws. A unified description of the velocity fields and the displacements is found. The unified equations of the field possess symmetry, and, due to this symmetry, the unified system is related to Laplace's equation. Formulations of boundary-value problems for a symmetric system of equations are proposed. The symmetry of the unified equations of the fields can be used in the numerical method of a boundary integral equation.

The problem of the formalism of fields in continua considered below has the following sources. Numerical calculations of the slow motions of a viscous fluid using a boundary integral equation are well known[1–6] in various problems of the motion of particles, bubbles and drops in a viscous fluid. The fundamental solution of the Stokes equations are usually used[1] to construct the boundary integral equation. Other possibilities for constructing a numerical method can give integral relations for harmonic functions. In order to use them, a relation between the solution of the Stokes equations and solutions of Laplace's equation is required. It is well known that hydrodynamic equations which do not take account of inertia are analogous to the equation of the theory of elasticity[7–17] for displacements at the equilibrium of a body. Existence theorems are known[7,14–18] for the main formulations of the problems. In investigations of planar problems of hydrodynamics and the theory of elasticity, the use of harmonic functions has been thoroughly developed using the apparatus of functions of a complex variable. In the case of spatial problems in the dynamics of a viscous fluid, a relation between the solutions of the Stokes equations and the solutions of Laplace's equation was established by Oberbeck[19,11,12] in the solution of the problem of the motion of an ellipsoid in a viscous fluid. The analogous Papkovich formula, which has also been considered by Neuber and Grodskii,[20–22,14,15,17] is known in the theory of elasticity. The vagueness with the number of functions in the formulae for the general solution led to a continuation of the investigations.[23,24]

Formulae for the general solution are also known in which the first and second derivatives of harmonic functions occur.[12,17]

Representations of the solutions of the Stokes equations in terms of harmonic functions (including several new representations of the solutions) together with an integral equation for a harmonic function have been used to calculate

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E-mail address: o.v.voinov@mtu-net.ru.

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viscous flow.[25] Unlike the approach to establishing the relation between the solutions of the classical field equations and the solutions of Laplace's equation, the problem of a new formulism for the two classical models of media in hydrodynamics and the theory of elasticity is studied below with the aim of describing the fields by harmonic functions. Asymmetric field equations are replaced by symmetric equations.

1. The unified equations of the velocity and displacement fields

For a slow flow of a viscous fluid, the velocity **u** with the components $u_{\rm i}$ satisfies the system of Stokes equations

$$\nabla_i p = \mu\Delta u_i, \quad \nabla_j u_j = 0; \quad i, j = 1, 2, 3 \tag{1.1}$$

where μ is the coefficient of dynamic viscosity, summation is carried out over repeated indices and the notation in a Cartesian system of coordinates x_1, x_2, x_3 is used. The displacements **u** with the components u_i in an elastic medium in statics satisfy the Navier equations

$$(1-2\sigma)\Delta u_i + \nabla_i \mathrm{div}\mathbf{u} = 0 \tag{1.2}$$

Poisson's ratio σ is expressed in terms of the Lamé coefficients: $2\sigma = \lambda/(\lambda+\mu)$.

It is well known that Eqs. (1.1) follow from Eqs. (1.2) when $\sigma = 1/2$.

We will show that the mathematical generality of the equations of fields without inertia in hydrodynamics and the theory of elasticity enables us to describe these fields in a unified manner. We rewrite systems (1.1) and (1.2) at a point x of a certain spatial domain V in the equivalent form

$$\mu\Delta u_i = \nabla_i p \tag{1.3}$$

$$\mu \mathrm{div}\mathbf{u} + (1-2\sigma)p = 0, \quad \mathbf{x} \in V \tag{1.4}$$

In the case of an elastic medium, μ is the corresponding Lamé coefficient. Here, the Navier equations of the theory of elasticity $\sigma \le 1/2$ and the Stokes equations of hydrodynamics ($\sigma = 1/2$) correspond to different values of Poisson's ratio.

We know that, in the case of a slow flow, the pressure satisfies Laplace's equation $\Delta p = 0$. We will use this in order to drop the equation for the incompressibility of a liquid (1.4). We now combine Laplace's equation for the pressure with the Stokes equation (1.3) into a system of equations for the velocity or displacement field

$$\Delta u_i = 2\nabla_i \Pi, \quad \Delta\Pi = 0, \quad \mathbf{x} \in V \tag{1.5}$$

Here, the pressure $p = 2\Pi\mu$. The system of equations (1.5) is the same for the velocities of a viscous incompressible fluid and displacements in an elastic body for any value of Poisson's ratio σ (Ref. 25)*.

In order to describe physical velocity or displacement fields by means of system (1.5), we specify a special condition on the boundary S of the domain V

$$\mathrm{div}\mathbf{u} + (1-2\sigma)2\Pi = 0, \quad \mathbf{x} \in S \tag{1.6}$$

2. Equivalence of the field formalisms

The following theorem for the equivalence of the two methods of describing a field holds for functions u_i and II which are single-valued and sufficiently smooth in V: the functions u_i are triply continuously differentiable and the function II is doubly continuously differentiable (this will be satisfied in the case of the usual assumptions concerning the smoothness of the field on a boundary of a domain and the smoothness of the boundary (for example, see Refs 7,14,16)).

Theorem. *System of equations (1.5) in the domain V and boundary condition (1.6) are equivalent to the system of equations (1.3) and (1.4).*

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* Preliminary account. See also: *Volnov OV*, Symmetric field equations in slow flows of a viscous incompressible fluid and elastic media, Moscow; 2006. Deposited in the All-Russia Institute for Scientific and Technical Information (VINITI) 31.05.06, No. 719-B2006.

Proof. It is sufficient to prove that the solutions of system of equations (1.5) with boundary condition (1.6) satisfies the original system (1.3) and (1.4), since the inverse assertion is obvious. It follows from system (1.5) that

$$\Delta\Lambda = 0, \quad \mathbf{x} \in V; \quad \Lambda = \operatorname{div}\mathbf{u} + (1 - 2\sigma)2\Pi$$

We now consider a Dirichlet problem for Laplace's equation in the function Λ in the domain V, setting up the trivial condition (1.6) on the boundary surface. By virtue of the uniqueness of this Dirichlet problem, a unique trivial solution $\Lambda \equiv 0$ exists which is identical to Eq. (1.4).

3. Symmetry of the unified system of field equations

System of equations (1.5) is invariant under a transformation which includes the four arbitrary harmonic functions

$$\Pi' = \Pi + \Phi, \quad u_i' = u_i + x_i\Phi + \varphi_i, \quad \Delta\Phi = 0, \quad \Delta\varphi_i = 0 \tag{3.1}$$

The field symmetry, characterized by the invariant transformation (3.1), is a property of system (1.5), that is, the symmetry of the field in an extended sense. It is important that (3.1) is not an invariant transformation for system (1.3), (1.4) which corresponds to systems of Stokes and Navier equations.

It is easy to prove that all of the solutions of the system of differential equations (1.5) are contained in the terms on the right-hand sides of the invariant transformation (3.1) and therefore, in the general case, the field is described by the same four equations

$$u_i = \Pi x_i + \upsilon_i, \quad \Delta\Pi = 0, \quad \Delta\upsilon_i = 0, \quad \mathbf{x} \in V \tag{3.2}$$

The harmonic functions Π and υ_i are arbitrary single-valued functions. We shall consider the special condition on the boundary S (1.6) together with the known boundary conditions for velocities (displacements) or stresses which are written down in the formulations of the boundary-value problems. The boundary conditions for all possible formulations of boundary-value problems for the Stokes and Navier equations can be used in the symmetric formalism.[7–16]

In the case of a fluid, the special condition (1.6) signifies the fluid incompressibility at the boundary surface S.

We emphasize that equations of the form (1.5) or formula (3.2) have, up to this time, not been considered as a system of differential field equations. When they have been considered[11,12,14,15,18] within the framework of known systems, with a differential equation of the incompressibility condition type (1.4) there is no symmetry of the system of equation and no simple description of the fields by harmonic functions.

4. Formulation of the boundary conditions for the symmetric field equations

According to the generalized Hooke's law, the stress tensor in an elastic medium has the following components

$$p_{ij} = 2\mu\varepsilon_{ij} + \delta_{ij}\lambda\varepsilon_{kk}, \quad \varepsilon_{ij} = (\nabla_i u_j + \nabla_j u_i)/2$$

where ε_{ij} are the components of the strain tensor and δ_{ij} is the Kronecker delta. Taking account of the definition of the quantity p in Eq. (1.4), we have

$$p_{ij} = 2\mu\varepsilon_{ij} - 2\sigma p\delta_{ij}$$

When $\sigma = 1/2$, the last relation also corresponds to the Navier-Stokes law for a viscous incompressible fluid if ε_{ij} are the components of the strain rate tensor.

Several types of boundary conditions can be used to calculate the flows of a viscous fluid. For instance, in the formulation of problems for system (1.5) (or (3.2)) on separate parts of the boundary surface S_0, S_1 and S_2, it is possible to specify the velocity vector (component by component)

$$u_i = u_i^0, \quad \mathbf{x} \in S_0 \subset S \tag{4.1}$$

the surface force vector

$$p_{ij}n_j = P_i, \quad \mathbf{x} \in S_1 \subset S \tag{4.2}$$

(**n** is the unit vector of the normal) and the normal velocity and the tangential stresses

$$\mathbf{u}\cdot\mathbf{n} = u_n, \quad p_{ij}n_j\tau_{1i} = P_{\tau 1}, \quad \mathbf{x}\in S_2\subset S \tag{4.3}$$

(τ_1 is one of two tangential unit vectors and the relation for $P_{\tau 2}$ is written in a similar way).

In the other boundary condition for the part of the surface $S_3\subset S$, conditions of periodicity of the velocity vector are specified (S_3 is the closed external part of the surface S). This condition enables us to describe periodic flows in the modelling of the motion of an infinite lattice of identical bodies in a viscous fluid.

In addition to the three scalar boundary conditions of one of the above mentioned types, we still specify the fourth condition (1.6).

In problems of the equilibrium of an elastic body, the parameter $\sigma\le 1/2$ in boundary condition (1.6). In these problems, the displacement vector (4.1) or surface force vector (4.2) can also be specified on separate parts of the boundary which correspond to the two basic formulations of the theory of elasticity. In a boundary value problem for the symmetric system of equations (1.5), there will be four boundary conditions of which three are conventional conditions and the fourth is the special condition (1.6).

Note that the order of a symmetric system of equations is higher than the order of the usual system of equations but, in spatial problems, this is of little importance when employing a numerical method using an integral equation.

5. Allowance for mass forces in the symmetric field equations

When account is taken of the body forces **X** with the components X_i, system (1.3), (1.4) has the form

$$\mu\Delta u_i + X_i = \nabla_i p, \quad \mu\,\mathrm{div}\,\mathbf{u} + (1-2\sigma)p = 0 \tag{5.1}$$

We replace system (5.1) in the domain V with the system

$$\Delta u_i + \frac{1}{\mu}X_i = 2\nabla_i\Pi, \quad \Delta\Pi = \frac{1}{\mu(4-4\sigma)}\nabla_j X_j, \quad \mathbf{x}\in V \tag{5.2}$$

and specify condition (1.6) on the boundary S of the domain V.

When account is taken of the external forces (5.2), the unified system of field equations is symmetric in the same sense as system (1.5) is without these forces being taken into account. System (5.2) is invariant under transformations (3.1), which includes the four arbitrary harmonic functions.

The symmetry and classical formalisms are equivalent. The following theorem holds: relations (5.2) and (1.6) are equivalent to system (5.1), which is analogous to the theorem in Section 2, in the case of the new formalism for the describing a field which takes account of the mass forces on the basis of (5.2) and (1.6). The unified system of equations (5.2) for the velocities and displacements is a system of field equations in an extended sense, and, when formulating the boundary-value problems, we therefore set the additional boundary condition.

We note the expressions for the general solution of Eqs. (5.2). Using any particular solution, for example, in the form of a Newtonian potential, the external forces can be eliminated from the second equation of (5.2). We denote the right-hand side of the last equation of (5.2) by $f(\mathbf{x})$. Then, the replacement

$$\Pi = \tilde{\Pi} - \frac{1}{4\pi}\int\limits_V \frac{f(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}dV \tag{5.3}$$

gives Laplace's equation $\Delta\bar{\Pi} = 0$. The contribution from the mass forces is eliminated from the first equation of (5.2) in a similar manner. An expression in terms of harmonic functions, which is analogous to expression (3.2), can then be written for the general solution of Eqs. (5.2).

We will assume that the mass forces have a potential U:$X_i = -\rho\nabla_i U$, where ρ is the density of the medium. Replacement of the quantities Π and U using the formulae

$$\Pi_* = \Pi + \frac{\rho}{\mu(4-4\sigma)}U, \quad W = \frac{1-2\sigma}{\mu(2-2\sigma)}\rho U \tag{5.4}$$

reduces relations (5.2) and (1.6) to the form

$$\Delta u_i - \nabla_i W = 2\nabla_i \Pi_*, \quad \Delta \Pi_* = 0 \tag{5.5}$$

$$\operatorname{div}\mathbf{u} + (1-2\sigma)2\Pi_* - W = 0, \quad \mathbf{x} \in S \tag{5.6}$$

Note that, by using the particular solution u_i^+ of the first equation of (5.5) when $\Pi_* = 0$, W can be eliminated by replacement of the variables u_i.

If the medium is incompressible ($\sigma = 1/2$), then $W \equiv 0$ and we obtain the system of equations (5.5) without the explicit effect of mass forces, where the conventional generalized pressure $p_* = 2\Pi_* \mu$.

Suppose the potential of the mass forces $U(\mathbf{x})$ is a harmonic function. After making the replacement $2\Pi_* + W = 2\tilde{\Pi}$, the potential W is eliminated from system (5.5) and, taking account of relations (3.2), it is easy to write the expression for the general solution.

6. Conclusion

A unified description of the spatial velocity fields of slow flows of a viscous incompressible fluid and the displacement fields in an elastic body in statics has been obtained. The unified system of differential field equations (1.5) or system (5.2), which takes account of mass forces, is invariant under a transformation associated with the representation of its general solution in terms of harmonic functions. There is no symmetry of the unified system in the case of systems of Stokes and Navier equations. The general solution of the symmetric system (1.5) is expressed by a linear form of four arbitrary harmonic functions.

In the formulation of boundary-value problems for a symmetric system of field equations, the usual three scalar conditions on the boundary surface and the special boundary condition (1.6) (in the case of a fluid, this is the incompressibility condition) are used. The symmetric system of equations is suitable for all problems concerning velocity or displacement fields without inertia for which a solution of the classical equations exists.

The symmetric field equations have been applied in a numerical method using a boundary integral equation.[25]

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